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in analytical geometry, of "imaginary intersections" or "circular points at infinity." No philosopher wishes to confute a mathematician because, in his technical language, the mathematician may assert that some "real" numbers are not "rational."

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NOTES ON THE CONSTRUCTION OF MAGIC SQUARES

OF ORDERS IN WHICH n IS OF THE GENERAL FORM $4p+2$.

It is well known that magic squares of the above orders, i. e., 6^2, 10^2, 14^2, 18^2, etc., cannot be made perfectly pandiagonal and ornate with the natural series of numbers.

Dr. C. Planck has however pointed out that this disability is purely arithmetical, seeing that these magics can be readily constructed as perfect and ornate as any others with a properly selected series of numbers.

In all of these squares n is of the general form $4p+2$, but they can be divided into two classes:

Class I. Where n is of the form $8p-2$, as 6^2, 14^2, 22^2 etc.

Class II. Where n is of the form $8p+2$, as 10^2, 18^2, 26^2 etc.

The series for all magics of Class I may be derived by making a square of the natural series 1 to $(n+1)^2$ and discarding the numbers in the middle row and column.

Thus, for a 6^2 magic the series will be:

1	2	3	—	5	6	7
8	9	10	—	12	13	14
15	16	17	—	19	20	21
—	—	—	—	—	—	—
29	30	31	—	33	34	35
36	37	38	—	40	41	42
43	44	45	—	47	48	49

The series for all magics of Class II may be made by writing a square of the natural numbers 1 to $(n+3)^2$ and discarding the numbers in the *three* middle rows and columns. The series for a 10^2 magic, for example, will be:

1	2	3	4	5	. . .	9	10	11	12	13
14	15	16	17	18	. . .	22	23	24	25	26
27	28	29	30	31	. . .	35	36	37	38	39
40	41	42	43	44	. . .	48	49	50	51	52
53	54	55	56	57	. . .	61	62	63	64	65
—	—	—	—	—	. . .	—	—	—	—	—
—	—	—	—	—	. . .	—	—	—	—	—
—	—	—	—	—	. . .	—	—	—	—	—
105	106	107	108	109	. . .	113	114	115	116	117
118	119	120	121	122	. . .	126	127	128	129	130
131	132	133	134	135	. . .	139	140	141	142	143
144	145	146	147	148	. . .	152	153	154	155	156
157	158	159	160	161	. . .	165	166	167	168	169

By using series as above described, pandiagonal magics with double-ply properties, or associated magics may be readily made either by the La Hireian method with magic rectangles, or by the path method as developed by Dr. C. Planck.

Referring now to the La Hireian method and using the 6^2 magic as a first example, the rectangles required for making the two auxiliary squares will necessarily be 2×3, and the numbers used therein will be those commonly employed for squares of the seventh order, i. e., $(6+1)^2$, with the middle numbers omitted thus:

1 2 3 — 5 6 7
0 7 14 — 28 35 42

It may be shown that a magic rectangle having an odd number of cells in one side, and an even number of cells in the other side is impossible with consecutive numbers, but with a series made as above it can be constructed without any difficulty, as shown in Figs. 1 and 2.

Two auxiliary squares may now be made by filling them with their respective rectangles. If this is done without forethought, a plain pandiagonal magic of the sixth order may result, but if attention is given to ornate qualities in the two auxiliaries, these features will naturally be carried into the final square. For example, by the arrangement of rectangles shown in Figs. 3 and 4 both auxiliaries are made magic in their six rows, six columns and twelve diagonals, and they are also 4-ply and 9-ply. Their complementary couplets are also harmoniously connected throughout in steps of

7	2	3
1	6	5

Fig. 1.

42	7	14
0	35	28

Fig. 2.

7	2	3	7	2	3
1	6	5	1	6	5
7	2	3	7	2	3
1	6	5	1	6	5
7	2	3	7	2	3
1	6	5	1	6	5

Fig. 3.

0	42	0	42	0	42
35	7	35	7	35	7
28	14	28	14	28	14
0	42	0	42	0	42
35	7	35	7	35	7
28	14	28	14	28	14

Fig. 4.

7	44	3	49	2	45
㊱	13	40	⑧	41	12
35	16	31	21	30	17
①	48	5	㊸	6	47
42	9	38	14	37	10
㉙	20	33	⑮	34	19

Fig. 5.

7	2	3	3	2	7
1	6	5	5	6	1
7	2	3	3	2	7
1	6	5	5	6	1
7	2	3	3	2	7
1	6	5	5	6	1

Fig. 6.

0	42	0	42	0	42
35	7	35	7	35	7
28	14	28	14	28	14
28	14	28	14	28	14
35	7	35	7	35	7
0	42	0	42	0	42

Fig. 7.

3, 3. These ornate features are therefore transmitted into the finished 6^2 magic shown in Fig. 5. If it is desired to make this square associated, that is with its complementary couplets evenly balanced around its center, it is only necessary to introduce the feature of association into the two auxiliary squares by a rearrangement of their magic rectangles as shown in Figs. 6, 7 and 8, the last figure being a pandiagonal associated magic.

7	44	3	45	2	49
36	13	40	12	41	8
35	16	31	17	30	21
29	20	33	19	34	15
42	9	38	10	37	14
1	48	5	47	6	43

Fig. 8.

The next larger square of Class I is 14^2, and it can be made with the natural series 1 to $(14+1)^2$ arranged in a square, discarding, as before, all the numbers in the central row and column.

The rectangles for this square will necessarily be 2x7 and the numbers written therein will be those ordinarily used for a square of the fifteenth order, $(14+1)^2$, with the middle numbers omitted, thus:

1	2	3	4	5	6	7	—	9	10	11	12	13	14	15
0	15	30	45	60	75	90	—	120	135	150	165	180	195	210

15	2	3	12	11	6	7
1	14	13	4	5	10	9

Fig. 9.

210	15	30	165	150	75	90
0	195	180	45	60	135	120

Fig. 10.

Simple forms of magic rectangles for the auxiliaries are shown in Figs. 9 and 10, but many other arrangements of the couplets will work equally well.

The smallest magic of Class II is 10^2, the series for which is given below. The rectangles used for filling the two auxilliaries of this square are 2x5, and they can be made with the numbers which would be commonly used for a square of the thirteenth order $(10+3)^2$ omitting the three middle numbers in each row thus:

1 2 3 4 5 . . . 9 10 11 12 13
0 13 26 39 52 . . . 104 117 130 143 156

Figs. 11 and 12 show these two rectangles with a simple arrangement of the numbers. The two auxiliaries and the finished 10^2 magic are given in Figs. 13, 14 and 15. Fig. 15 is magic in its ten rows, ten columns and twenty diagonals. It is also 4-ply and 25-ply. Like the 6^2 magic, this square can also be associated by changing the disposition of the magic rectangles in the auxiliaries.

13	2	11	4	5
1	12	3	10	9

Fig. 11.

156	13	130	39	52
0	143	26	117	104

Fig. 12.

13	2	11	4	5	13	2	11	4	5
1	12	3	10	9	1	12	3	10	9
13	2	11	4	5	13	2	11	4	5
1	12	3	10	9	1	12	3	10	9
13	2	11	4	5	13	2	11	4	5
1	12	3	10	9	1	12	3	10	9
13	2	11	4	5	13	2	11	4	5
1	12	3	10	9	1	12	3	10	9
13	2	11	4	5	13	2	11	4	5
1	12	3	10	9	1	12	3	10	9

Fig. 13.

The above examples will suffice to explain the general construction of these squares by the La Hireian method with magic rectangles. It may however be stated that although the series previously described for use in building these squares include the lower numerical values, there are other series of higher numbers which will produce equivalent magic results.

0	156	0	156	0	156	0	156	0	156
143	13	143	13	143	13	143	13	143	13
26	130	26	130	26	130	26	130	26	130
117	39	117	39	117	39	117	39	117	39
104	52	104	52	104	52	104	52	104	52
0	156	0	156	0	156	0	156	0	156
143	13	143	13	143	13	143	13	143	13
26	130	26	130	26	130	26	130	26	130
117	39	117	39	117	39	117	39	117	39
104	52	104	52	104	52	104	52	104	52

Fig. 14.

13	158	11	160	5	169	2	167	4	161
144	25	146	23	152	14	155	16	153	22
39	132	37	134	31	143	28	141	30	135
118	51	120	49	126	40	129	42	127	48
117	54	115	56	109	65	106	63	108	57
1	168	3	166	9	157	12	159	10	165
156	15	154	17	148	26	145	24	147	18
27	142	29	140	35	131	38	133	36	189
130	41	128	43	122	52	119	50	121	44
105	64	107	62	113	53	116	55	114	61

Fig. 15.

The following table illustrates another rule covering the selection of numbers for all magic squares of these orders.

ORDER OF SQUARE	NATURAL SERIES	DISCARDING NUMBERS IN
6th	1 to $(6+1)^2$	the middle row and column.
10th	1 to $(10+3)^2$	the 3 middle rows and columns.
14th	1 to $(14+5)^2$	the 5 middle rows and columns.
18th	1 to $(18+7)^2$	the 7 middle rows and columns.
22nd	1 to $(22+9)^2$	the 9 middle rows and columns.
26th	1 to $(26+11)^2$	the 11 middle rows and columns.
		and so forth.

These figures show that this rule is equivalent to taking the numbers of the natural series $\left(\frac{3n-4}{2}\right)^2$ and omitting the central $\frac{n-4}{2}$ rows and columns. In comparing the above with the rules previously given, for which we are indebted to Dr. C. Planck, it will be seen that in cases of magics larger than 10^2 it involves the use of unnecessarily large numbers.

The numerical values of the ply properties of these squares are naturally governed by the dimensions of the magic rectangles used in their construction. Thus the rectangle of the 6^2 magic (Fig. 5) is 2x3, and this square is 2^2-ply and 3^2-ply. The rectangle of the 10^2 magic being 2x5, the square may be made 2^2-ply and 5^2-ply, and so forth.

The formation of these squares by the Path method which has been so ably developed by Dr. C. Planck,[1] may now be considered. The first step is to rearrange the numbers of the given series in such a cyclic order or sequence, that each number being written consecutively into the square by a well defined rule or path, the resulting magic will be identical with that made by the La Hireian method, or equivalent thereto in magic qualities. Starting, as before, with the 6^2 magic, the proper sequence of the first six numbers is found in what may be termed the "continuous diagonal" of its magic rectangle. Referring to Fig. 1, this sequence is seen to be 1, 2, 5, 7, 6, 3, but it is obvious that there may be as many different sequences as there are variations in the magic rectangles.

The complete series given on page 304 must now be rearranged in its *lines and columns* in accordance with the numerical sequence of the first six numbers as above indicated. To make this arrangement quite clear, the series given on p. 304 is reproduced in Fig. 16, the numbers written in circles outside the square showing the numerical order of lines and columns under rearrangement. Fig. 17 shows the complete series in new cyclic order, and to construct a square directly therefrom, it is only necessary to write these numbers consecutively along the proper paths. Since the square will be pandiagonal *it may be commenced anywhere,* so in the present example we will place 1 in the fourth cell from the top in the first column, and will use the paths followed in Fig. 5 so as to reproduce that square. The paths may be written $\begin{vmatrix} 3, 2 \\ 4, 3 \end{vmatrix}$ and since we can always write

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[1] "The Theory of Path Nasiks," by C. Planck, M.A., M.R.C.S., published by A. T. Lawrence, Rugby, England.

	①	②	⑥	③	⑤	④
①	1	2	3	5	6	7
②	8	9	10	12	13	14
⑥	15	16	17	19	20	21
③	29	30	31	33	34	35
⑤	36	37	38	40	41	42
④	43	44	45	47	48	49

Fig. 16.

	①	②	③	④	⑤	⑥
①	1	2	5	7	6	3
②	8	9	12	14	13	10
③	29	30	33	35	34	31
④	43	44	47	49	48	45
⑤	36	37	40	42	41	38
⑥	15	16	19	21	20	17

Fig. 17.

$-(n-a)$ instead of a, we may write this $\begin{vmatrix} 3, 2 \\ -2, 3 \end{vmatrix}$. This only means that the numbers in the first column of Fig. 17 (which may be termed the *leading numbers*) are to be placed in order along the path (3, 2), as in the numbers enclosed in circles in Fig. 5; and then starting from each cell thus occupied, the remaining five numbers in each of the six rows of Fig. 17 are to be written along the path (−2, 3). It will be seen that this is equivalent to writing the successive rows of Fig. 17 intact along the path (−2, 3), or (3, −2) and using a "break-step" (1, −1), as in Fig. 18 where the first break-step is shown with an arrow. The break-step is always given by summing up the coordinates; thus, the paths here being $\begin{vmatrix} 3, 2 \\ -2, 3 \end{vmatrix}$, by summing the columns we get (1, 5), that is (1, −1). The resulting square is, of course, identical with Fig. 5.

As previously stated, this square being pandiagonal, it may be commenced in any of its thirty-six cells, and by using the same methods as before, different aspects of Fig. 5 will be produced. Also, since by this method complementary pairs are always separated by a step ($n/2$, $n/2$), any of the thirty-six squares thus formed may be made associated by the method described in *The Monist,* Vol. XX, No. 3, page 443, under the heading "Magic Squares by Complementary Differences,"[2] viz., Divide the square into four quarters as shown in Fig. 19; leave A untouched, reflect B, invert C and

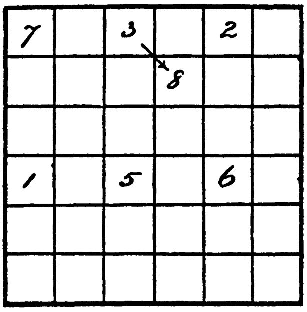

Fig. 18.

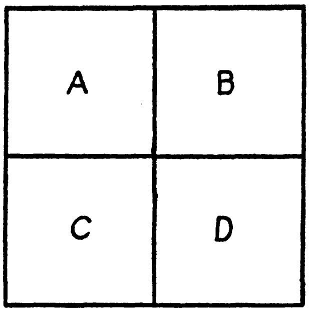

Fig. 19.

1	2	3	4	9	13	12	11	10	5
14	15	16	17	22	26	25	24	23	18
27	28	29	30	35	39	38	37	36	31
40	41	42	43	48	52	51	50	49	44
105	106	107	108	113	117	116	115	114	109
157	158	159	160	165	169	168	167	166	161
144	145	146	147	152	156	155	154	153	148
131	132	133	134	139	143	142	141	140	135
118	119	120	121	126	130	129	128	127	122
53	54	55	56	61	65	64	63	62	57

Fig. 20.

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[2] Errata in this article: p. 440, footnote, and p. 443, fourth line from top of page, instead of "for all orders $=4n+2$" read "for orders wherein n is of the general form $4p+2$."—Page 44, last line, for "order $8n$" read "this class."

reflect and invert D. For this concise and elegant method of changing the relative positions of the complementary couplets in a square we are indebted to Dr. Planck.

The next square in order is 10². The series of numbers used is given on page 305 and their rearrangement in proper cyclic order for direct entry may be found as before in the continuous diagonal of its magic rectangle. The sequence shown in Fig. 11 is 1, 2, 3, 4, 9, 13, 12, 11, 10, 5, and the complete rearrangement of the series in accordance therewith is given in Fig. 20. Various 10² magics may be made by using this series with different paths. The paths $\begin{vmatrix} 5, 4 \\ -4, 5 \end{vmatrix}$ will produce Fig. 15, and $\begin{vmatrix} 5, 2 \\ 2, 5 \end{vmatrix}$ will make Fig. 21, which is equivalent to Fig. 15 in its ornate features.

13	160	2	161	11	169	4	158	5	157
27	140	38	139	29	131	36	142	35	133
117	56	106	57	115	65	108	54	109	63
144	23	155	22	146	14	153	25	152	16
130	43	119	44	128	52	121	41	122	50
1	166	12	165	3	157	10	168	9	159
39	134	28	135	37	143	30	132	31	141
105	62	116	61	107	53	114	64	113	55
156	17	145	18	154	26	147	15	148	24
118	49	129	48	120	40	127	51	126	42

Fig. 21.

21	2	3	4	17	16	15	8	13
1	20	19	18	5	6	7	14	9

Fig. 22.

These squares and all similarly constructed larger ones of these orders may be changed to the form of association wherein the complementary couplets are evenly balanced around the center of the square, by the method previously explained. It will be unnecessary to prolong the present article by giving any examples of larger squares of this class, but the simple forms of magic rectangles for 18² and 22² and 26² magics, shown in Figs. 22, 23 and 24, may be of some assistance to those who desire to devote further study to these interesting squares.[3]

23	2	21	4	19	6	17	8	9	10	13
1	22	3	20	5	18	7	16	15	14	11

Fig. 23.

29	2	27	4	25	6	23	8	9	20	11	18	13
1	28	3	26	5	24	7	22	21	10	19	12	17

Fig. 24.

W. S. Andrews.
L. S. Frierson.

A NEW THEORY OF INVENTION.

A Russian engineer, P. K. von Engelmeyer of Moscow (Petersburger Chaussée 42), has published a little book on invention and its significance in our industrial life under the title *Der Dreiakt* (Berlin, Carl Heymann's Verlag) in which he claims that man is not only a political being (ζῷον πολιτικόν) as Aristotle claims, but also and mainly a technical being (ζῷον τεχνικόν), and he means it in the same sense in which Franklin called man a "tool-making animal."[1]

Mr. Engelmeyer defines technique as the art of reproducing artificially or intentionally certain desired phenomena (p. 17) and he calls attention to the fact that we are surrounded by the products of invention. Our clothes, the light and heat in our houses, our mode of traveling, in short, all that is called culture and civilization has

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[3] More generally, if p, q are relative primes, the square of order pq will be magic on its pq rows, pq columns and $2pq$ diagonals, and at the same time p^2-ply and q-^{2}ply, if it be constructed with the paths $\begin{vmatrix} p, q \\ q, p \end{vmatrix}$, and the period be taken from the continuous diagonal of the magic rectangle $p \times q$. The limitations are dictated by the magic rectangle. Evidently p and q must both be > 1, and consecutive numbers must fail if the order is $\equiv 2 \pmod{4}$; in all other cases consecutive numbers will suffice. C. P.

[1] See the author's *The Philosophy of the Tool*, p. 1.